

# $U(2)$ instantons and leptons

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## Abstract

Anomalous quantisation of instantons allows non-trivial (anti) self-dual configurations to exist for pure  $U(2)$  gauge theory in four-dimensional Euclidean space-time and to be used as ground state of the model. Six left fermions together with six right fermions and one complex scalar doublet describe the largest possible vacuum. This structure is just enough to represent the leptons and Higgs boson in the Standard Model.

The aim of the present note is to construct a vacuum for pure  $U(2)$  gauge field in four-dimensional Euclidean space-time  $E^4$  using instantons. Quantisation in the vicinity of classical solution is a common technique in Quantum Field Theory. When the classical solution/vacuum is not unique but depends on some parameters the standard path integral over all fields is replaced by a functional integral over the quantum field (deviation from the vacuum) plus ordinary integrals over all parameters on which the vacuum configuration depends [1]. In our case the problem is even more subtle because we do not know the vacuum explicitly. Instead of this we know that it satisfies some equations which are different of the equations of motion — the (anti) self duality equations. The problem is consider in Ref.[2] for  $U(1)$  gauge field and here we deal with the  $U(2)$  case with the following action

$$\mathbf{A} = -\frac{1}{4g'^2} \int F_{\mu\nu}^\alpha F_{\mu\nu}^\alpha. \quad (1)$$

Here  $\alpha = 0, 1, 2, 3$ .  $F_{\mu\nu}^0$  is the field strength tensor for the  $U(1)$  potential  $A_\mu^0$  and  $F_{\mu\nu}^a$ ,  $a = 1, 2, 3$  are the field strength tensor components for the  $SU(2)$  potential  $A_\mu^a$ .<sup>1</sup>

The idea is to find a class of classical solutions of the action (1) around which to quantise the theory. As the (anti) self-dual fields trivially satisfy the equations of motion we shall use them to construct the largest possible vacuum. The ground state thus obtained is not unique and a proper physical parametrisation of it has to be found. As a first step we have to rid ourselves off the gauge degrees of freedom in the vacuum. So, a gauge fixing of the instantons is required. Then we rewrite the (anti-) self dual conditions and gauge fixing conditions in a close form which allows direct particle interpretation. Two types of mutually commuting quaternions are used for this purpose. We refer to these quaternions as  $e$ - and  $\xi$ -quaternions respectively with  $\{e_\mu\}$  and  $\{\xi^\alpha\}$  as their quaternion units. The  $e$ -quaternions are connected to the space-time while the  $\xi$ -quaternions are connected to the internal space. Thus, having a gauge potential  $A_\mu^\alpha$  we construct out of it four  $e$ -quaternion functions  $\mathcal{A}^\alpha = e_\mu A_\mu^\alpha$  and one bi-quaternion function  $\mathbb{A} = \xi^\alpha \mathcal{A}^\alpha$ . We shall use also the  $e$ -conjugated to  $\mathcal{A}$  and  $\mathbb{A}$  functions which we denote  $\bar{\mathcal{A}}^\alpha$  and  $\bar{\mathbb{A}}$  respectively. All together

$$\begin{aligned} \mathbb{A} &= \xi^\alpha e_\mu A_\mu^\alpha \\ \bar{\mathbb{A}} &= \xi^\alpha \bar{e}_\mu A_\mu^\alpha. \end{aligned} \quad (2)$$

Here  $\bar{e}_\mu$  is the quaternion conjugated to  $e_\mu$ . Note that there is no  $\xi$ -quaternion conjugation in the definition of  $\bar{\mathbb{A}}$ . Two conjugated first order,  $e$ -quaternion

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<sup>1</sup>See the Appendix for details on most formulae.

valued differential operators  $\mathcal{D}$  and  $\bar{\mathcal{D}}$  will be used as well

$$\begin{aligned}\mathcal{D} &= e_\mu \partial_\mu \\ \bar{\mathcal{D}} &= \bar{e}_\mu \partial_\mu.\end{aligned}\tag{3}$$

A  $e$ -quaternion function  $\mathcal{F}$  is called Fueter (quaternion) analytic if it satisfies the equation

$$\mathcal{D}\mathcal{F} = 0\tag{4}$$

and Fueter anti-analytic if it satisfies the equation

$$\bar{\mathcal{D}}\mathcal{F} = 0.\tag{5}$$

The Fueter analytic and anti-analytic functions are essential for the construction of the  $U(1)$  nontrivial instanton vacuum. In the  $U(2)$  case the instanton equations are a nonlinear version of the Fueter (anti) analyticity conditions, namely, the equation

$$(\bar{\mathcal{D}} + \frac{g}{2}\bar{\mathbb{A}})\mathbb{A} = 0\tag{6}$$

describes self-dual field configuration in a certain gauge, while the equation

$$(\mathcal{D} + \frac{g}{2}\mathbb{A})\bar{\mathbb{A}} = 0\tag{7}$$

describes anti self-dual field in the same gauge. Different gauge conditions can be obtained adding an arbitrary  $\xi$ -quaternion function to eqs.(6,7). (Arbitrary function of  $\mathcal{A}^\alpha$  and  $\bar{\mathcal{A}}^\alpha$  works fine. A function depending on the gauge fields derivatives can be used as well but in this case we have to be sure that the gauge condition is nontrivial and fixes the gauge.)

We would like to find solutions of eqs.(6,7) which admit free particle interpretation. Two such solutions are direct consequence of the  $U(1)$  instanton solution proposed in Ref.[2] and correspond to the two possible  $U(1)$  subgroup of  $U(2)$  — the  $U(1)$  group in the decomposition  $U(2) = U(1) \times SU(2)$  and the Cartan subgroup of the  $SU(2)$ . Each of these solutions is parametrised by a couple of Fueter analytic and anti-analytic functions. Additional parameters are needed in order to take into account some features of the model. First, we have a global  $SO(3)$  invariance in the  $SU(2)$  ground solution due to the ambiguity in the choice of the Cartan subgroup. The best way to count corresponding degrees of freedom is to introduce a basis in the adjoint representation. This gives a factor of three. Second, note that the two types of instantons can be freely combined producing particles with two charges. In this case another  $Z_2$  symmetry emerges. The symmetry is connected to the relative sign of the two charges. This gives a factor of two. Thus we end up

with vacuum described by seven couples of Fueter analytic and anti-analytic functions. These functions, after anomalous quantisation [3], correspond to seven right and seven left Wail fermions.

Let us finish with a selected list of some of the open questions which have to be answered in order to identify the vacuum just constructed with the lepton generations.

- We have excess of fermions. A possible way out of this problem is to quantise one of the Fueter analytic — anti-analytic couples normally. This possibility is based on the following considerations: Note that for quaternions the equation

$$\square Q = 0 \tag{8}$$

because of the identity  $\mathcal{D}\bar{\mathcal{D}} = \square$  has two solutions, namely

$$\begin{aligned} \mathcal{D}Q_1 &= 0 \\ \bar{\mathcal{D}}Q_2 &= 0. \end{aligned}$$

Therefore, we can parametrise one instanton and one anti instanton with a complex scalar doublet. However in this case we cannot gain stability as a consequence of the instanton anomalous statistics. Thus we have replaced one problem with another. The benefit of the above construction is that now we have a candidate for the Higgs boson. In this case our vacuum consists of one scalar doublet and twelve Wail fermions which coincides at least with the type and the number of matter fields in the Standard Model.

- There is no difference in our treatment between left and right chiral fermions. The question why the left ones are  $SU(2)$  doublets and the rights are singlets cannot be answered.
- No answer what are the particular values of the charges.
- Only three of the considered vacuum solutions are stable, i.e. we have too few stable particles.

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# Appendix

## Quaternions

The quaternion number  $\mathcal{Q}$  is defined as follows:

$$\mathcal{Q} = e_\mu q_\mu \quad (9)$$

where  $q_\mu$ ,  $\mu = 0, \dots, 3$  are four real numbers and  $e_\mu$  are the quaternion units:

$$\begin{aligned} e_0 e_0 &= 1 \\ e_i e_0 &= e_0 e_i = e_i \\ e_i e_j &= -\delta_{ij} + \epsilon_{ijk} e_k. \end{aligned} \quad (10)$$

The representation of the  $e$ -quaternion units we are using here is

$$e_0 = 1, \quad e_k = -i\sigma_k, \quad k = 1, 2, 3 \quad (11)$$

where  $\sigma_k$  are the Pauli matrices. The  $\xi$ -quaternion units will be represented as real  $4 \times 4$  matrices ( $2 \times 2$  block matrices)

$$\xi^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \xi^1 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \xi^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \quad (12)$$

where  $I = -i\sigma_2$ . These two representations of the quaternion units together give well defined representation of  $\mathbb{Q}^2$ .

Quaternion conjugation:

$$\overline{(\mathcal{Q})} \equiv \bar{\mathcal{Q}} = q_0 - e_i q_i. \quad (13)$$

Field strength tensor

$U(1)$  strength tensor

$$F_{\mu\nu}^0 = \partial_\mu A_\nu^0 - \partial_\nu A_\mu^0. \quad (14)$$

$SU(2)$  strength tensor components

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g[A_\mu, A_\nu]^a, \quad a = 1, 2, 3. \quad (15)$$

$U(1)$  charge —  $g'$ .

$SU(2)$  charge —  $g'g$ .

Self-duality condition

$$F_{\mu\nu}^\alpha = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}^\alpha, \quad \alpha = 0, 1, 2, 3 \quad (16)$$

( $\epsilon$  is the totally anti symmetric tensor.) For each  $\alpha$  these are three independent conditions. In the  $U(1)$  case they are

$$\partial_0 A_i^0 - \partial_i A_0^0 - \epsilon_{ijk}\partial_j A_k^0 = 0 \quad (17)$$

and for  $SU(2)$  they are

$$\partial_0 A_i^a - \partial_i A_0^a - \epsilon_{ijk}\partial_j A_k^a + g\epsilon^{abc}\left(A_0^b A_i^c - \frac{1}{2}\epsilon_{ijk}A_j^b A_k^c\right) = 0. \quad (18)$$

Here we have used  $[A_\mu, A_\nu]^a = \epsilon^{abc}A_\mu^b A_\nu^c$ .

Some algebra

Eq.(6) is equivalent to the following equations

$$\begin{aligned} 0 &= \partial_\mu A_\mu^0 + \frac{g}{2}((A^0)^2 - (A^a)^2) \\ 0 &= \partial_\mu A_\mu^a + gA^0 \cdot A^a \\ 0 &= \partial_0 A_i^0 - \partial_i A_0^0 - \epsilon_{ijk}\partial_j A_k^0 \\ 0 &= \partial_0 A_i^a - \partial_i A_0^a - \epsilon_{ijk}\partial_j A_k^a + g\epsilon^{abc}(A_0^b A_i^c - \frac{1}{2}\epsilon_{ijk}A_j^b A_k^c) \end{aligned} \quad (19)$$

and so, it describes self dual gauge field (last two of eqs.(19)) in a gauge which is a variant of the Lorentz gauge (first two equations). Analogously, eq.(7) is equivalent to the following equations

$$\begin{aligned} 0 &= \partial_\mu A_\mu^0 + \frac{g}{2}((A^0)^2 - (A^a)^2) \\ 0 &= \partial_\mu A_\mu^a + gA^0 \cdot A^a \\ 0 &= \partial_0 A_i^0 - \partial_i A_0^0 + \epsilon_{ijk}\partial_j A_k^0 \\ 0 &= \partial_0 A_i^a - \partial_i A_0^a + \epsilon_{ijk}\partial_j A_k^a + g\epsilon^{abc}(A_0^b A_i^c + \frac{1}{2}\epsilon_{ijk}A_j^b A_k^c) \end{aligned} \quad (20)$$

and it describes anti self dual field in the same gauge as in eq.(19). The different lines in eqs.(19,20) correspond to the coefficients (up to sign) for the quaternion units 1,  $\xi^a$ ,  $e_i$  and  $\xi^a e_i$ , so in both cases the gauge conditions are pure  $\xi$ -quaternions while the (anti) self-dual conditions are imaginary  $e$ -quaternions.

Simple instanton solution I

Let  $\mathcal{A}^a = 0 \quad \forall a$  and  $\mathcal{A}^0 \neq 0$ . In this case eq.(6) reduces (after suitable change of the gauge) to Fueter anti analyticity condition (eq.(5)) for  $\mathcal{A}^0$  while

eq.(7) goes to Fueter analyticity condition (eq.(4)). The anomalous quantised solutions of equations (4,5) are massless fermions [3].

Simple instanton solution II

Pick up an index  $a$ , say  $a = \mathbf{a}$ . Let  $\mathcal{A}^\alpha = 0 \ \forall \alpha \neq \mathbf{a}$  and  $\mathcal{A}^{\mathbf{a}} \neq 0$ . Again eq.(6) reduces to eq.(5) for  $\mathcal{A}^{\mathbf{a}}$  and eq.(7) reduces to eq.(4).

There is a global  $SO(3)$  covariance for solution II: If  $\mathcal{A}^a$  is a solution, so is the field  $\mathcal{A}'^a = U^{ab}\mathcal{A}^b$  where  $U^{ab}$  is  $SO(3)$  matrix.

## References

- [1] J.-L. Gervais and B. Sakita, Phys. Rev. **D 10** (1975) 2943.
- [2] M.N. Stoilov, CEJP 5(4) 2007 507515.
- [3] D.Ts. Stoyanov, J. Phys. A: Math. Gen. **25** (1992) 4245.